J. Fluid Mech. (1966), *vol.* 25, *part* 1, *pp.* 121–142
Printed in Great Britain

On the damping of internal gravity waves in a continuously stratified ocean

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(Received 5 March 1965 and in revised form 1 November 1965)

The problem studied here is that of the attenuation of internal waves through turbulent mixing in a weakly and exponentially stratified fluid. The equations are linearized and it is assumed that the action of turbulence can be parametrically represented by eddy mixing coefficients and that the influence of bottom friction is restricted to a thin bottom boundary layer. The simple case where there is no rotation and only one component to the stratification is first examined in detail, and the modifications caused by introducing rotation and a second component are subsequently investigated. Subject quantitatively to the choice made for the eddy coefficients, but qualitatively not strongly dependent on that choice, the following conclusions are drawn: (i) very short internal waves (length $< 100\,$m) are strongly damped in basins of all depths; (ii) long internal waves or seiches in shallow seas (depth $\simeq 100\,$m) will not last more than a few cycles as free oscillations; (iii) the attenuation rate for long internal tides is small enough that these should be observable very far from the coasts, but large enough to exclude the possibility of oceanic standing wave systems; (iv) for very long internal waves the damping is predominantly due to the effect of bottom friction, and the attenuation rate becomes almost independent of the actual form of the stratification present in the fluid.

1. Introduction

Rattray (1957) has already pointed out the importance of knowing the dissipation rate of internal waves in order to correctly assess theories attempting to explain their origin. An exclusively coastal generation theory would, for example, be inconsistent with a strong damping factor and observation of large internal waves in the high seas. The work of Rattray, like that of his predecessors in this field (Harrison 1908; Ekman 1931), deals with the frictional attenuation of oscillations of the interface between two superimposed fluids of slightly different densities. Such waves are found to be damped at a rate proportional to the square root of the eddy viscosity, the amplitude of long waves being reduced by a half in a distance of the order of 1000 km (Rattray 1957).

Since it may not be justifiable to approximate everywhere the oceanic density field by a discontinuous stratification, it is interesting to extend this investigation to continuously stratified fluids. A step in that direction has already been taken by Krauss (1964), who finds that the damping rate is proportional to the first power of the eddy viscosity but numerically smaller than that found by Rattray. The present work also takes into account the dissipative effect of mixing processes, considers the influence of rotation and examines in more detail the assumption involved in the use of eddy diffusion and viscosity coefficients to represent the action of turbulence on internal waves.

2. On eddy coefficients

Most flows of geophysical interest are turbulent, and in order to obtain a solution of the hydrodynamic equations one often assumes that the non-linear interaction between turbulent and mean flow (the influence of the Reynolds stresses) is analogous to the linear molecular mixing processes and can be represented by 'eddy' viscosity and diffusion coefficients. The main justification of this approximation is of course that it overcomes the mathematical difficulties involved in the solution of non-linear equations.

The analogy between the Reynolds and viscous stresses is however a very gross one, and this subterfuge can only give an idea of the order of magnitude of the amount of energy extracted by the turbulence from the mean flow. One should then not attribute too much significance to those results which depend critically on the value of the eddy parameters, and even those results which are almost independent of the value of the parameters are still subject to the assumption as to the form of the interaction.

The eddy coefficients should be functions of the intensity and the spatial distribution of the turbulence as well as of the scale of the mean flow with which it interacts. The spatial dependence may however fall if the turbulent field is homogeneous. With a wave motion as mean flow, the interaction with a turbulent field and its dependence on wavelength is clearly put forward by this qualitative argument of Groen (1954): 'turbulent eddies which are small in comparison with the wavelength cause an internal friction, and consequently a loss of ordered energy, or a decay of the waves. Eddies however which are large compared to the wavelength only cause local changes of phase velocity and group velocity. Short waves which have passed through such an eddy may, after leaving its sphere of action, have undergone refraction, but will not, on the average, have lost energy thereby.' It is clear that whether an eddy is considered large or small depends on the length of a particular wave, and the action of turbulence, represented parametrically by an eddy viscosity, will be a function of the wave length, L.

It is convenient, when examining the interaction of waves and turbulence, to divide the turbulent field into two parts: 'internally' generated turbulence, produced by the waves themselves, and 'externally' generated turbulence, which exists independently of the wave motion. Bowden (1960) and Groen (1954) have investigated the influence of turbulent fields of the first and second type respectively on surface gravity waves. On the basis of a dimensional argument, Bowden uses an eddy viscosity K proportional to the wave amplitude A and the wave velocity C so that $K = \text{const.}\, CA$. On the basis of the qualitative argument given above, Groen uses Richardson's $\frac{4}{3}$ power law (Richardson &

Stommel 1948) to write $K = \text{const.}\, L^{\frac{4}{3}}$. Comparison of their predicted damping rates with attenuation characteristics of long swell are however inconclusive in both cases, so that the choice of the form of the eddy coefficients is not critically tested.

These concepts are easily extended to internal gravity waves. The general effect of a stable stratification $\rho_0(z)$ is to reduce the intensity of the turbulence, particularly in the vertical direction. Turbulent energy is as usual dissipated by molecular friction and is also, through molecular diffusion, transferred to gravitational potential energy: the density gradients are eroded, which, for a stable stratification, leads to an increase of potential energy. It may also happen, for two-component stratifications, that potential energy is extracted from the density field, provided that the two components have gradients of opposite signs and their diffusivities are not identical (Walin 1964). The turbulent field may still be considered horizontally homogeneous, but because of the reduced intensity in the vertical direction two sets of eddy coefficients must now be used, one describing turbulent exchange in the horizontal (K_h, of order 10^3 to $10^8\,\text{cm}^2\,\text{sec}^{-1}$) and the other in the vertical direction (K_v, of order 10 to $10^3\,\text{cm}^2\,\text{sec}^{-1}$).

If the turbulence is of the 'internally' generated type, the dimensional argument of Bowden may still be made; the proportionality constant will now presumably depend on the Richardson number. A study of the mixing effects of internal waves in the ocean, by Glinskii & Boguslavskii (1963), based on a slightly different eddy viscosity (directly proportional to the square of the amplitude, inversely to the period), gives reasonable results for the heat transfer below the thermocline, so that at least no contradictory results arise from using such an eddy viscosity. The qualitative argument of Groen will hold for internal as well as for surface waves. Since the turbulence is no longer three-dimensionally homogeneous, the $L^{\frac{4}{3}}$ dependence of the eddy coefficients is no longer a reasonable assumption. Once the use of eddy viscosity parameters has been resolved upon, these arguments can be extended to introduce eddy diffusion coefficients for the diffusion of heat and salt.

An averaging process must now be devised by means of which the turbulent field can be formally separated from the mean flow (here, the field of internal waves) and a Reynolds equation derived. Let us first describe the problem.

Consider a layer of incompressible fluid of thickness H, of infinite lateral extent, rotating with angular velocity Ω about an axis perpendicular to its equilibrium surface. Let us choose right-handed co-ordinates $\mathbf{x} = (x, y, z)$ rotating with the fluid and such that $z = 0$ at the upper surface and increases downwards. The corresponding velocity components are denoted by $\mathbf{u} = (u, v, w)$. The fluid is, on a time average, at rest and is endowed with a weak density stratification, $\rho_0(z)$, maintained by unspecified but adequate sources of heat and salt. This zero-order state is described by vanishing velocities and hydrostatic pressure:

$$\mathbf{u}_0 = 0, \quad p_0 = \rho_0 g z + \text{const.} \tag{1}$$

Superimposed upon this is a horizontally homogeneous and nearly stationary field of 'external' turbulence characterized by velocity and density fluctuations $\mathbf{u}_t(\mathbf{x}, t)$, $\rho_t(\mathbf{x}, t)$, with $\rho_t \ll \rho_0$. Because of homogeneity, time averages are independent of horizontal co-ordinates, and because of the nearly stationary behaviour all non-zero space averages, are only slowly varying functions of time. Since this turbulent field is decaying, no sources need be postulated to maintain it; it is only assumed that the decay time-constant is much longer than the period of the waves.

Propagating in this field of turbulence in the positive x-direction are flat-crested internal waves of angular frequency ω and wave-number k, and of small enough amplitude that we may neglect squares of velocities as well as density disturbances due to the waves as compared to the main density field. The turbulent field will be separated from the waves by averaging in a horizontal direction perpendicular to the direction of propagation (i.e. along the crests, in the y-direction). Since the turbulent variables are random in phase and homogeneous in space, the y-average will leave only the wave field. Denoting the averaged wave field by overscored variables and the total turbulent perturbations (including the extra perturbations caused by turbulent transport of water particles endowed with wave velocities) by primed variables, the total field is given by

$$\left.\begin{aligned} \mathbf{U} &= \overline{\mathbf{u}}+\mathbf{u}', \\ \rho &= \rho_0+\overline{\rho}+\rho', \\ P &= p_0+\overline{p}+p'. \end{aligned}\right\} \tag{2}$$

All primed quantities have zero y-average; all overscored variables have zero time-average.

Substitution of (2) in the Navier–Stokes equations and averaging in the y-direction gives, to first order in the overscored variables, a Reynolds equation; in tensor notation,

$$\frac{\partial \overline{u}_i}{\partial t} - 2\epsilon_{ijk}\overline{u}_j\Omega_k - \frac{\overline{\rho}}{\rho_0} g\delta_{i3} + \frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x_i} = \frac{\partial}{\partial x_j}\left[\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}\right]. \tag{3}$$

It is now that an eddy viscosity coefficient is introduced to linearize the problem. Subject to the limitations outlined at the beginning of this section, one writes, in formal analogy to the molecular friction term.

$$\frac{\partial}{\partial x_j}(-\overline{u_i'u_j'}) = \frac{\partial}{\partial x_r}\left[K_m'\frac{\partial \overline{u}_i}{\partial x_r}\right]. \tag{4}$$

The conservation equation for some diffusive property $\overline{T}$ (temperature, say) corresponding to (3) will be

$$\frac{\partial \overline{T}}{\partial t} + \overline{w}\frac{dT_0}{dz} = \frac{\partial}{\partial x_j}\left[\kappa\frac{\partial \overline{T}}{\partial x_j} - \overline{T'u_j'}\right]. \tag{5}$$

The eddy heat diffusion coefficients K_h' are introduced in the same manner as the viscosity coefficients:

$$\frac{\partial}{\partial x_j}(-\overline{T'u_j'}) = \frac{\partial}{\partial x_r}\left[K_h'\frac{\partial \overline{T}}{\partial x_r}\right]. \tag{6}$$

The above analysis holds for a field of 'externally' generated turbulence. One sees, nevertheless, that if 'internally' generated turbulence is of random phase (just as likely to be a positive as a negative perturbation), averaging in the y-direction will allow the fields to be separated as in (2) and derivation of Reynolds and diffusion equations. If the damping rate of the waves is small their turbulence-producing capacity is nearly constant and the resulting turbulent field is also nearly stationary. Horizontal homogeneity cannot be assumed and three different eddy coefficients are required. These coefficients will have an entirely different dependence upon the wave parameters as those introduced to represent the 'external' turbulent field, but no new notation will be used and the two sets lumped under the same symbols.

3. The wave equation

The fluid has already been assumed incompressible, so that the density will be a function of temperature (T) and salinity (S) only. For simplicity, $\rho(\mathbf{x}, t)$ is assumed to be linear in these two variables; for the steady field,

$$\rho_0(z) = \rho_0(0) - \alpha_1 T_0(z) + \alpha_2 S_0(z), \tag{7}$$

and for the wave field,

$$\bar{\rho}(\mathbf{x}, t) = -\alpha_1 \bar{T}(\mathbf{x}, t) + \alpha_2 \bar{S}(\mathbf{x}, t). \tag{8}$$

The coefficients α_1 and α_2 are positive. Although this linearization is not a bad approximation to the actual salinity dependence, with $\alpha_2 = 8{\cdot}149 \times 10^{-3}\,\mathrm{g\,cm^{-3}}$ per °C (Fofonoff 1961), the temperature variation lends itself to a linear approximation only for a narrow range of temperatures at the time, and α_1 is left unspecified, but of order $10^{-4}\,\mathrm{g\,cm^{-3}/°C}$.

Eddy coefficients for momentum, heat and salt diffusion will in general be different from each other, and will now be written so as to include the corresponding molecular coefficients: for example,

$$K_m = K'_m + \nu.$$

The coefficients K_m, etc., are taken to be spatially uniform; they are still free to depend on the wave parameters. Although it is reasonable to assume that for a horizontally homogeneous field of turbulence the mixing does not depend on position, some restriction must be put on the density field to extend this to the vertical direction. The vertical intensity of the turbulence will depend, other things being equal, on the degree of stratification of the fluid. It is clear then that if there is to be some sense in taking eddy coefficients independent of z, the fluid must at least be uniformly stratified. This is so when $d\rho_0/dz$ is constant or, for weak stratifications ($\Delta\rho_0 \ll \rho_0$), when the stability parameter, $\rho_0^{-1}\, d\rho_0/dz$, is a constant. Our attention will then be restricted to weak exponential density fields, of uniform stability; this is not a bad approximation for many situations where the density increases smoothly and gradually between top and bottom, but it will not apply if any pycnocline is present.

Adding an extra index (r) to the eddy coefficients to take their anisotropy into account (K_{mr}, for example, is the eddy viscosity coefficient in the r-direction), the linearized momentum equation becomes

$$\left(\frac{\partial}{\partial t} - K_{mr}\frac{\partial^2}{\partial x_r^2}\right)\bar{u}_i - 2\epsilon_{ijk}\bar{u}_j\,\Omega_k - \frac{\bar{\rho}}{\rho_0}g\delta_{i3} + \frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x_i} = 0. \tag{9}$$

The y-averaged continuity equation is

$$\partial \bar{u}_i / \partial x_i = 0. \tag{10}$$

Independent equations for the conservation of heat and salt are, with eddy coefficients K_{hr} and K_{sr} respectively,

$$(\partial/\partial t - K_{hr}\, \partial^2/\partial x_r^2)\, \bar{T} + \bar{w}\, dT_0/dz = 0, \tag{11}$$

$$(\partial/\partial t - K_{sr}\, \partial^2/\partial x_r^2)\, \bar{S} + \bar{w}\, dS_0/dz = 0. \tag{12}$$

Equations (7) to (12) completely describe the dynamics of the linearized system, for weak stratifications. All variables but one can be eliminated and a wave equation for the vertical velocity derived.

First taking the curl of (9) and selecting the z-component of the resulting vorticity equation, we have

$$(\partial/\partial t - K_{mr}\, \partial^2/\partial x_r^2)\, \bar{\zeta} - 2\Omega\, \partial \bar{w}/\partial z = 0, \tag{13}$$

where $\bar{\zeta}$ is the z-component of the vorticity. Now taking twice the curl of (9) and again selecting the z-component, we find

$$\left(\frac{\partial}{\partial t} - K_{mr}\frac{\partial^2}{\partial x_r^2}\right)\nabla^2 \bar{w} + 2\Omega \frac{\partial \bar{\zeta}}{\partial z} - g\nabla_h^2 \frac{\bar{\rho}}{\rho_0} = 0. \tag{14}$$

The vector symbols for the Laplacian and its horizontal component (∇_h^2) have been used, for brevity and in spite of notational purity.

By operating on (14) with $(\partial/\partial t - K_{mr}\, \partial^2/\partial x_r^2)$ and using (13), $\bar{\zeta}$ is eliminated:

$$\left[\left(\frac{\partial}{\partial t} - K_{mr}\frac{\partial^2}{\partial x_r^2}\right)^2 \nabla^2 + 4\Omega^2 \frac{\partial^2}{\partial z^2}\right]\bar{w} - g\left(\frac{\partial}{\partial t} - K_{mr}\frac{\partial^2}{\partial x_r^2}\right)\nabla_h^2 \frac{\bar{\rho}}{\rho_0} = 0. \tag{15}$$

Substituting for $\bar{\rho}$ in terms of its components $\bar{T}$ and $\bar{S}$ through (8) and eliminating these by successively operating on (15) with

$$\left(\frac{\partial}{\partial t} - K_{hr}\frac{\partial^2}{\partial x_r^2}\right) \quad \text{and} \quad \left(\frac{\partial}{\partial t} - K_{sr}\frac{\partial^2}{\partial x_r^2}\right)$$

and using (11) and (12) the following wave equation in $\bar{w}$ alone results:

$$\begin{aligned}&\left[\left(\frac{\partial}{\partial t} - K_{mr}\frac{\partial^2}{\partial x_r^2}\right)^2 \nabla^2 + 4\Omega^2 \frac{\partial^2}{\partial z^2}\right]\left(\frac{\partial}{\partial t} - K_{hr}\frac{\partial^2}{\partial x_r^2}\right)\left(\frac{\partial}{\partial t} - K_{sr}\frac{\partial^2}{\partial x_r^2}\right)\bar{w} \\ &\quad + \frac{g}{\rho_0}\left[-\alpha_1 \frac{dT_0}{dz}\left(\frac{\partial}{\partial t} - K_{sr}\frac{\partial^2}{\partial x_r^2}\right) + \alpha_2 \frac{dS_0}{dz}\left(\frac{\partial}{\partial t} - K_{hr}\frac{\partial^2}{\partial x_r^2}\right)\right]\left(\frac{\partial}{\partial t} - K_{mr}\frac{\partial^2}{\partial x_r^2}\right)\nabla_h^2 \bar{w} = 0.\end{aligned} \tag{16}$$

For a one-component stratification, this equation is formally identical to that applying to the study of thermal convection in a fluid heated from below (Chandrasekhar 1961). The differences between the two problems lie in the sign of the density gradient and in the anisotropy of the present mixing coefficients.

The boundary conditions are taken as for the convection problem. Krauss (1965) has indeed shown that approximating the free surface dynamical boundary condition by $\bar{w}(x, y, 0; t) = 0$ has for main effect the loss of surface waves and a small change in the eigenvalues of the first internal mode. We are not here interested in surface waves and will also sacrifice some accuracy to the simplicity of this reduced boundary condition. Moreover, not only the upper surface, but also the bottom of the ocean will be considered incapable of supporting stress. The results will of course not be directly applicable to the ocean, but the simplicity of the solution and the closed analytical form of the results allow better understanding of the influence of the various physical parameters. The solution is thus developed for an ocean with a free-slip bottom. The inadequacy of the lower boundary condition will be supplemented by a boundary-layer analysis to evaluate the influence of bottom friction.

On both bounding surfaces then the vertical velocity and the density perturbations vanish:

$$\overline{w}(\mathbf{x},t) = \overline{T}(\mathbf{x},t) = \overline{S}(\mathbf{x},t) = 0 \quad \text{at} \quad z = 0, H, \tag{17}$$

and so do the stresses and torques

$$\partial^2\overline{w}/\partial z^2 = \partial\overline{\zeta}/\partial z = 0 \quad \text{at} \quad z = 0, H. \tag{18}$$

With the help of equations (11)–(17) the boundary conditions can be expressed in terms of the vertical velocity alone:

$$\partial^{2N}\overline{w}/\partial z^{2N} = 0 \quad \text{for} \quad N = 0, 1, \ldots, 4 \quad \text{at} \quad z = 0, H. \tag{19}$$

4. Wave solution

Waves of the form

$$\overline{w} = W(z)\, e^{\omega t - ikx} \tag{20}$$

are postulated as solutions. They propagate with a given wave-number k in the positive x-direction. The depth-dependent function $W(z)$ and the unknown complex frequency eigenvalue ω are to be determined from the stratification and the boundary conditions.

The following non-dimensional variables, operators and parameters are now introduced.

$$z' = z/H; \quad D = \partial/\partial z; \quad a = kH.$$

$$K_m = \frac{1}{3}\sum_{j=1}^{3} K_{mj}, \text{ and similarly for } K_h \text{ and } K_s.$$

$$\sigma = \omega H^2/K_m.$$

$$p_{fr}(K_{mr}, K_{hr}, K_{sr}) = K_m^{-1}(K_{mr}\delta_{f3} + K_{hr}\delta_{f1} + K_{sr}\delta_{f2} + K_m\delta_{f0}), \text{ where } f = 0, 1, 2, 3.$$

$$p_f = p_{fr}(K_m, K_h, K_s); \quad p = \frac{1}{3}\sum_{f=1}^{3} p_f. \tag{21}$$

$$\Gamma_1 = \frac{d(\frac{1}{2}\rho_0(0) - \alpha_1 T_0)/dz}{\frac{1}{2}\rho_0(0) - \alpha_1 T_0}; \quad \Gamma_2 = \frac{d(\frac{1}{2}\rho_0(0) + \alpha_2 S_0)/dz}{\frac{1}{2}\rho_0(0) + \alpha_2 S_0}.$$

$$\gamma = (\tfrac{1}{2}\rho_0(0) + \alpha_2 S_0)/(\tfrac{1}{2}\rho_0(0) - \alpha_1 T_0).$$

$$\Gamma = \rho_0^{-1} d\rho_0/dz = (\Gamma_1 + \gamma\Gamma_2)/(1+\gamma).$$

$$R = gH^4\Gamma/(K_m^2 p), \text{ a Rayleigh number.}$$

$$R_f = gH^4\Gamma_f/(K_m^2 p_f), \text{ a partial Rayleigh number; } f = 1, 2.$$

$$\Theta = 4\Omega^2 H^4/K_m^2, \text{ a Taylor number.}$$

Substituting the postulated wave solution (20) into (16) we have, in the notation of (21):

$$\{[\sigma-(p_{33}D^2-p_{31}a^2)]^2(D^2-a^2)+\Theta D^2\}[\sigma-(p_{13}D^2-p_{11}a^2)][\sigma-(p_{23}D^2-p_{21}a^2)]W$$
$$-a^2(1+\gamma)^{-1}\{R_1p_1[\sigma-(p_{23}D^2-p_{21}a^2)]+\gamma R_2\,p_2[\sigma-(p_{13}D^2-p_{11}a^2)]\}$$
$$\times[\sigma-(p_{33}D^2-p_{31}a^2)]\,W=0. \quad (22)$$

The boundary conditions (19) become

$$D^{2N}W=0, \quad N=0,1,\ldots,4 \quad \text{at} \quad z'=0,1. \quad (23)$$

For a weak exponential stratification Γ_1 and Γ_2 are constants and $\gamma\simeq 1$, so that equation (22) has constant coefficients and solutions satisfying boundary conditions (23) are of the form

$$W(z')=\text{const.}\sin(n\pi z'), \quad \text{with} \quad n=1,2,\ldots. \quad (24)$$

Substituting (24) into (22) we obtain an algebraic characteristic equation for σ. Let us write

$$\eta_f=p_{f3}n^2\pi^2+p_{f1}a^2, \quad f=0,1,2,3, \quad (25)$$

which gives the characteristic equation a more compact form:

$$[(\sigma+\eta_3)^2\eta_0+\Theta n^2\pi^2](\sigma+\eta_1)(\sigma+\eta_2)+a^2(1+\gamma)^{-1}[R_1p_1(\sigma+\eta_2)+\gamma R_2\,p_2(\sigma+\eta_1)]$$
$$\times(\sigma+\eta_3)=0. \quad (26)$$

Special cases of this equation will be studied.

5. Non-rotating fluid

(a) The effect of the temperature field

Let us first retain only the temperature field. Equation (26) reduces to the quadratic

$$\sigma^2+(\eta_1+\eta_3)\,\sigma+\eta_1\eta_3+a^2Rp/\eta_0=0, \quad (27)$$

which has complex roots

$$\sigma=-\tfrac{1}{2}(\eta_1+\eta_3)\pm i[a^2Rp/\eta_0-\tfrac{1}{4}(\eta_1-\eta_3)^2]^{\frac{1}{2}}. \quad (28)$$

The real part of σ is always negative and the waves are always damped. The magnitude of the exponential damping coefficient is proportional to the wave-number squared multiplied by the appropriate diffusion coefficient and the influences of momentum and heat diffusion are directly additive. The functional dependence is the same as that obtained by Krauss (1964) when only viscosity acts, and (28) reduces to his result when $\eta_1=0$, $p_{31}=p_{33}$. The imaginary part of σ also contains a term arising from the mixing and frictional processes, so that the frequency of damped waves is smaller than that of undamped ones. Moreover, if this second term becomes so large that

$$(\eta_1-\eta_3)^2\,\eta_0/(4a^2Rp)\geqslant 1, \quad (29)$$

only critically damped motions will be found.

We can characterize the damping rate by a time constant, τ, in which the wave amplitude decreases by a factor $(1/e)$:

$$\tau=2H^2/[(\eta_1+\eta_3)\,K_m]. \quad (30)$$

For propagating waves it is however more meaningful, since the velocity is also affected by mixing and friction, to compare with the wavelength ($L = 2\pi/k$) the distance x_e travelled by the wave at the phase velocity during the time τ:

$$x_e/L = \frac{a}{\pi(\eta_1+\eta_3)}\left(\frac{Rp}{\eta_0}\right)^{\frac{1}{2}}\left(1-\frac{(\eta_1-\eta_3)^2\eta_0}{4a^2Rp}\right)^{\frac{1}{2}}. \tag{31}$$

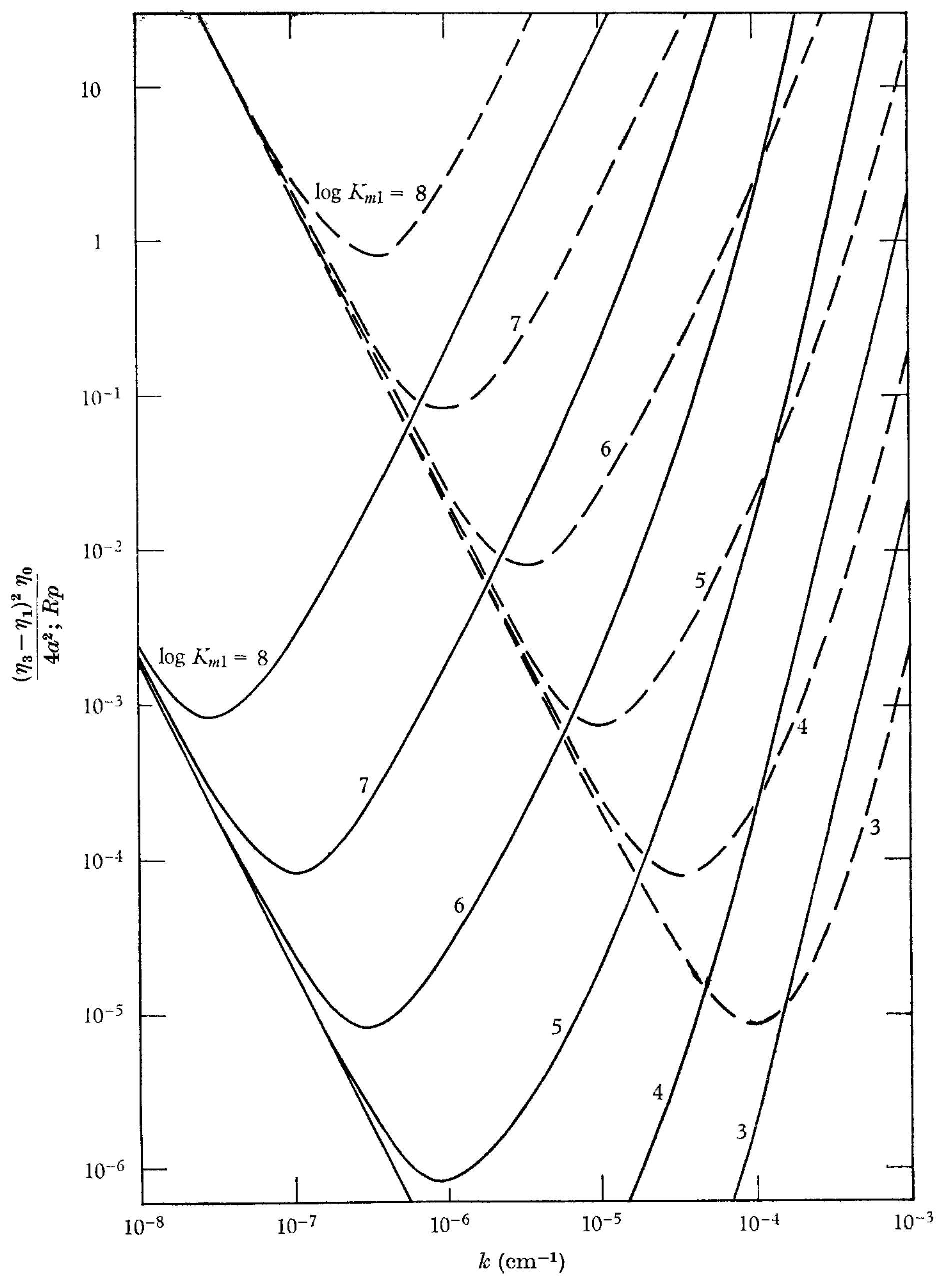

FIGURE 1. The influence of dissipative processes on the frequency of internal waves. $\Omega = 0$, $\Gamma_2 = 0$, $n = 1$. Continuous lines, $H = 1000$ m; broken lines, $H = 100$ m.

This quantity is also, for small dissipation rates, the ratio of the average energy content of the wave to the amount of energy lost per cycle through turbulent interaction. x_e/L is then equal to the Q of the oscillating system.

The period lengthening is directly related to the ratio $(\eta_1-\eta_3)^2\eta_0/(4a^2Rp)$, the dependence of which on some of the wave parameters is illustrated in figure 1. The dependence of the relative damping length, x_e/L, on wave-number, total

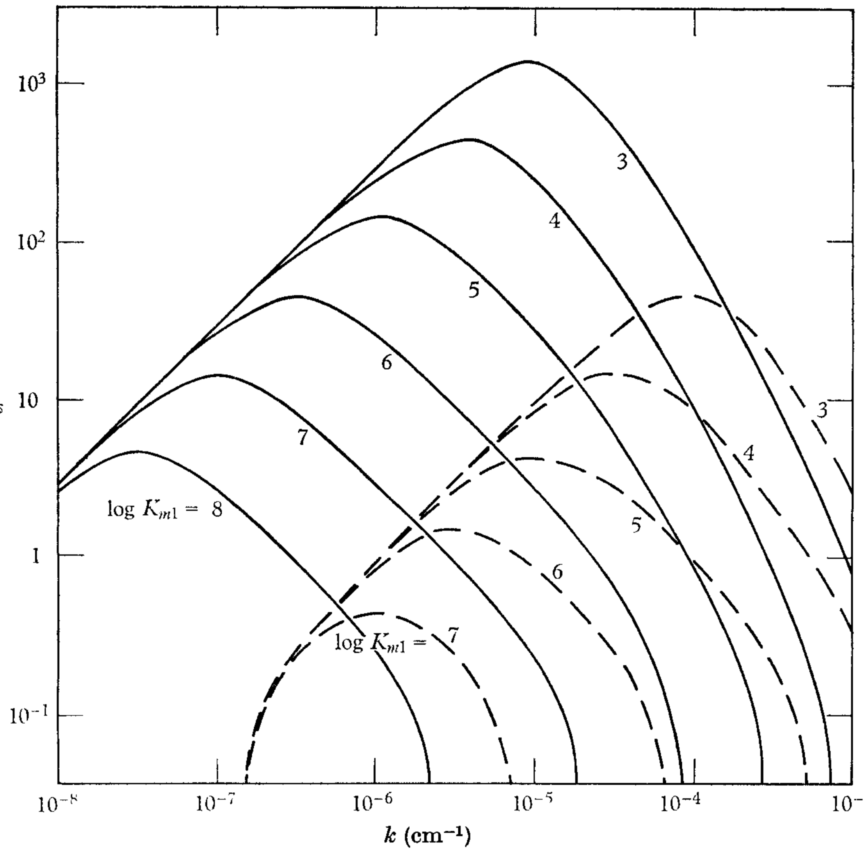

FIGURE 2. The relative damping length for $\Omega = 0$, $\Gamma_2 = 0$, $n = 1$. Continuous lines, $H = 1000$ m; broken lines, $H = 100$ m.

depth, eddy viscosity and vertical mode number is shown in figures 2 and 3. In these examples the values $\Delta\rho_0/\rho_0 = 10^{-3}$, $K_{m3} = 10^2\,\mathrm{cm^2\,sec^{-1}}$ have been used; η_1/η_3 is taken as 0·1 in all numerical examples.

The damping and period increase are in general dependent on the least well-known parameter, K_{m1}; a few general features may however be noted which are independent of the magnitude of the eddy coefficients: (1) The relative damping length and period increase present a maximum and a minimum, respectively, at intermediate wavelengths; the position of these extrema depends on K_{m1}. Such a wavelength dependence would act as a filter if K_{m1} was independent of wavelength: very far from their region of generation one would observe mostly internal waves with lengths near that corresponding to the extrema. When

K_{m1} depends on the wavelength in the general way outlined in §2, this effect becomes of little importance. It may, however, be of some consequence in the purely laminar case (LeBlond 1965). (2) The damping is more rapid when K_{m1} is larger, but the influence of K_{m1} becomes smaller at long wavelengths. (3) The

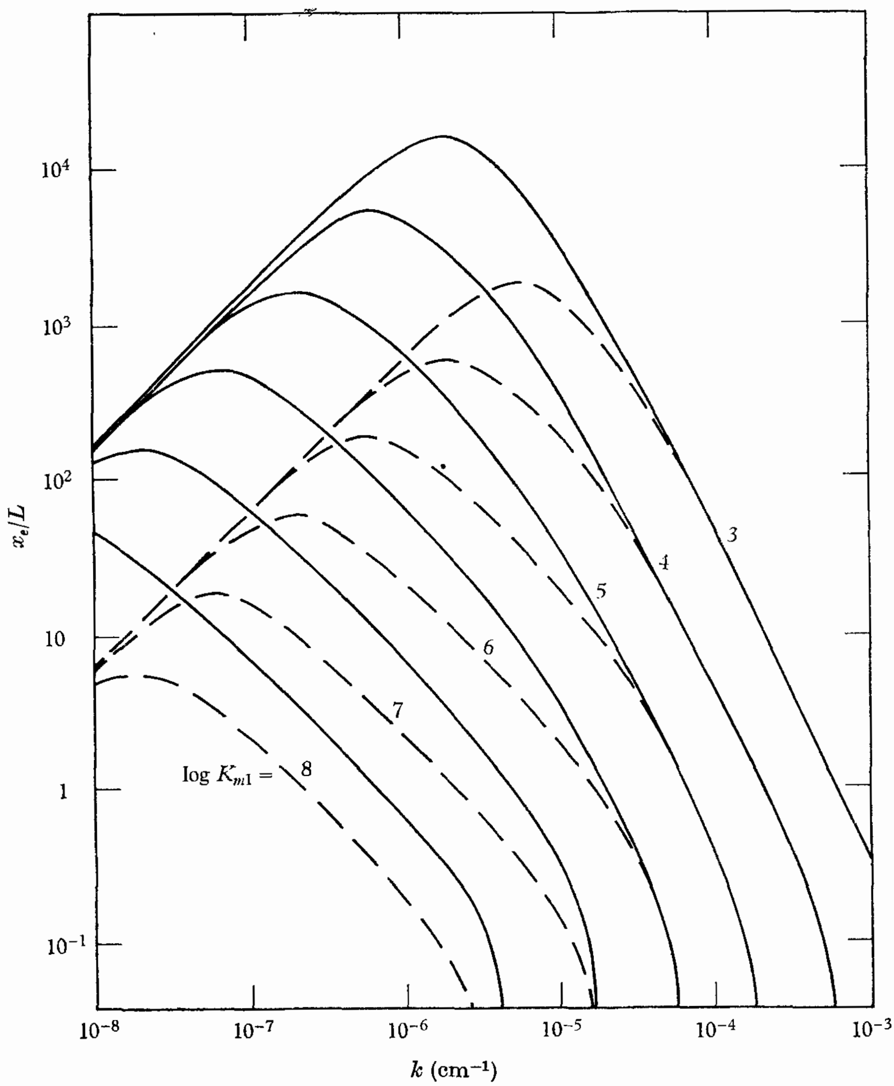

FIGURE 3. The relative damping length for $\Omega = 0$, $\Gamma_2 = 0$, $H = 5000$ m. Continuous lines, $n = 1$; broken lines, $n = 3$.

dependence of the relative damping length on the depth of the water layer is in opposite directions for long and short waves; in the first case, (31) depends on H mostly through its numerator, in the second case through its denominator. (4) The damping rate and the frequency lowering increase with the vertical mode number, n; for very large n, the relative damping length is proportional to n^{-3}.

This effect will, however, not manifest itself until the n variation plays the dominant role in η_f: as long as $n^2\pi^2 \ll a^2$, that is in the direction of large depths and short waves, the damping rate will not depend much on n. In other words, the vertical mode number becomes important only when vertical mixing predominates. (5) The relative damping length increases directly as the square root of the relative magnitude of the stratification, $\Delta\rho_0/\rho_0$, except very near the cut-off points.

Let us next see how the addition of a second component to the stratification (salinity) modifies this simple picture.

(b) The effect of the temperature and salinity fields

Still with $\Omega = 0$, but now with Γ_2 non-zero, the characteristic equation becomes

$$\sigma^3 + (\eta_1+\eta_2+\eta_3)\,\sigma^2 + \left[(\eta_1\eta_2+\eta_2\eta_3+\eta_1\eta_3) + \frac{a^2Rp}{\eta_0}\right]\sigma + \left[\eta_1\eta_2\eta_3 + \frac{a^2R_2p_2\gamma}{(1+\gamma)\,\eta_0}(\epsilon\eta_2+\eta_1)\right] = 0. \tag{32}$$

The symbol ϵ has been introduced to represent the relative contributions of the temperature and salinity fields to the total stratification:

$$\epsilon = \frac{R_1p_1}{\gamma R_2p_2} = -\frac{d(\alpha_1T_0)/dz}{d(\alpha_2S_0)/dz}. \tag{33}$$

The domain of ϵ is divided into three subdomains by the gravitational stability condition

$$R_2(\epsilon+1) \geq 0. \tag{34}$$

When R_1 and $R_2 > 0$, $\epsilon > 0$; when $R_1 < 0, R_2 > 0$, $-1 < \epsilon < 0$; and when $R_1 > 0$, $R_2 < 0$, $\epsilon < -1$.

Equation (32) will lead to solutions of oscillatory form when it has complex roots, that is, when its cubic discriminant is positive. This condition is satisfied except when

$$[(9sq/2-s^3)-(s^2-3q)^{\frac{3}{2}}] < \frac{27}{2}\left[\eta_1\eta_2\eta_3 + \frac{a^2R_2\,p_2\gamma}{(1+\gamma)\,\eta_0}(\epsilon\eta_2+\eta_1)\right] < [(9sq/2-s^3)+(s^2-3q)^{\frac{3}{2}}], \tag{35}$$

with $s = \eta_1+\eta_2+\eta_3$, $q = \eta_1\eta_2+\eta_1\eta_3+\eta_2\eta_3$.

For $\eta_1 = \eta_2$, the two components of the stratification are dynamically undistinguishable; for $\eta_2 = 0$, the second component does not diffuse and the effect of its presence will be a reduction of the damping rate; when $\epsilon = 0$, we have the same problem as in § 5*a* with salinity instead of temperature. More generally, the wave-number where transition from internal waves to critically damped motions occurs is now a function of η_2/η_1. The cut-off point is only slightly changed in the direction of small wave-numbers, but bands where internal waves can exist now appear at wave-numbers larger than the cut-off wave-number. Figure 4 shows as an example the dependence of the position of upper wave-number frequency cut-off on ϵ and η_2/η_1 for $\Delta\rho_0/\rho_0 = 10^{-3}$, $K_{m3} = 10^2\,\mathrm{cm^2\,sec^{-1}}$, $K_{m1} = K_{m2} = 10^8\,\mathrm{cm^2\,sec^{-1}}$, $n = 1$, and $H = 10^5\,\mathrm{cm}$.

The ratio of the molecular diffusivity of salt to that of heat is 10^{-3}, but nothing definite is known about the ratio of their eddy diffusivities in the ocean. Figure 4 shows that if $\eta_2 \neq \eta_1$ there results in general an extension towards shorter wavelengths or the possibility of existence of internal waves. Under the restriction to be introduced later on the maximum value of the eddy coefficients, the upper cut-off point will in any case never be closely approached.

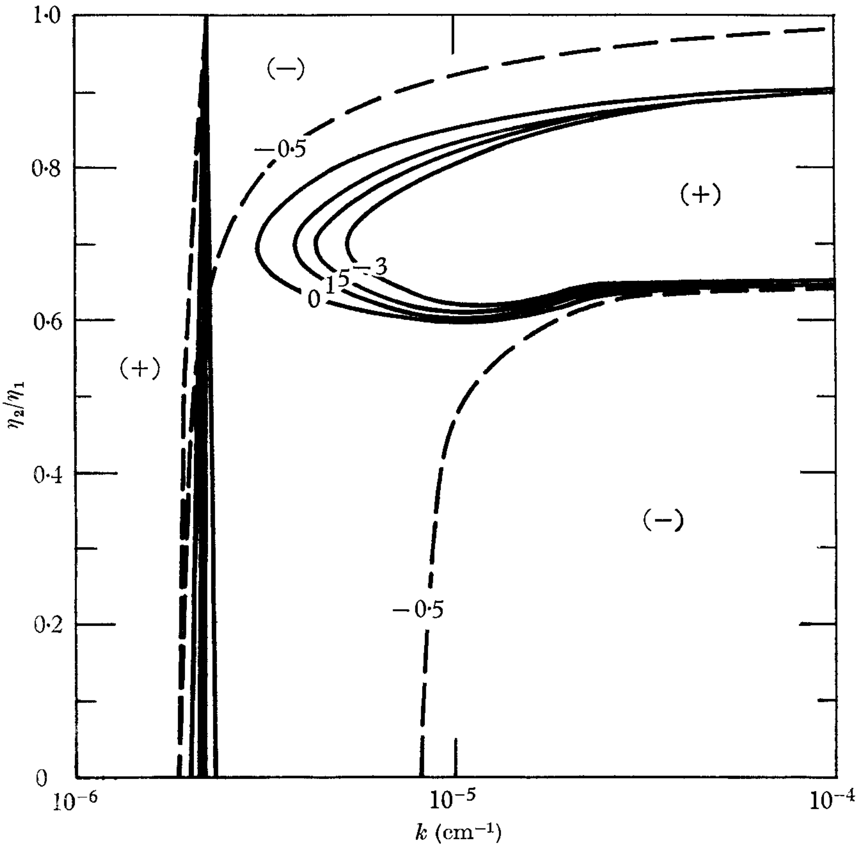

FIGURE 4. The loci of vanishing frequency for a two-component stratification. Broken line, $\epsilon = -0{\cdot}5$; continuous lines, from left to right, $\epsilon = 0, 1, 5, -3$. $(+)$ denotes regions where internal waves may exist; $(-)$ denotes regions where only critically damped motions are possible.

Another interesting property of two-component stratifications has been pointed out by Stern (1960) and by Walin (1964). Because of a difference in the diffusivities of the two components, a gravitationally stable density field may turn out to be dynamically unstable. The criterion derived by Walin for instability to wave motions (that the real part of σ be $\geqslant 0$, its imaginary part non-zero) is, in the notation used here,

$$\frac{a^2R_2p_2\gamma}{(1+\gamma)\eta_0}\frac{\epsilon(\eta_1+\eta_3)+(\eta_2+\eta_3)}{(\eta_1+\eta_2+\eta_3)(\eta_1\eta_2+\eta_2\eta_3+\eta_1\eta_3)-\eta_1\eta_2\eta_3} \leqslant -1. \tag{36}$$

There exists a band of wave-numbers where the system is unstable to wave motion whenever

$$R_2[\epsilon(\eta_1+\eta_3)+(\eta_2+\eta_3)] \leqslant 0. \tag{37}$$

Equations (34) and (37) are compatible (assuming $\eta_1 > \eta_2$) only when the temperature increases with depth ($\Gamma_1 < 0$), ϵ being restricted then to the range

$$-1 \leqslant \epsilon \leqslant -(\eta_3+\eta_2)/(\eta_3+\eta_1). \tag{38}$$

For the molecular diffusivities, this range is

$$-1 \leqslant \epsilon \leqslant -0{\cdot}910. \tag{39}$$

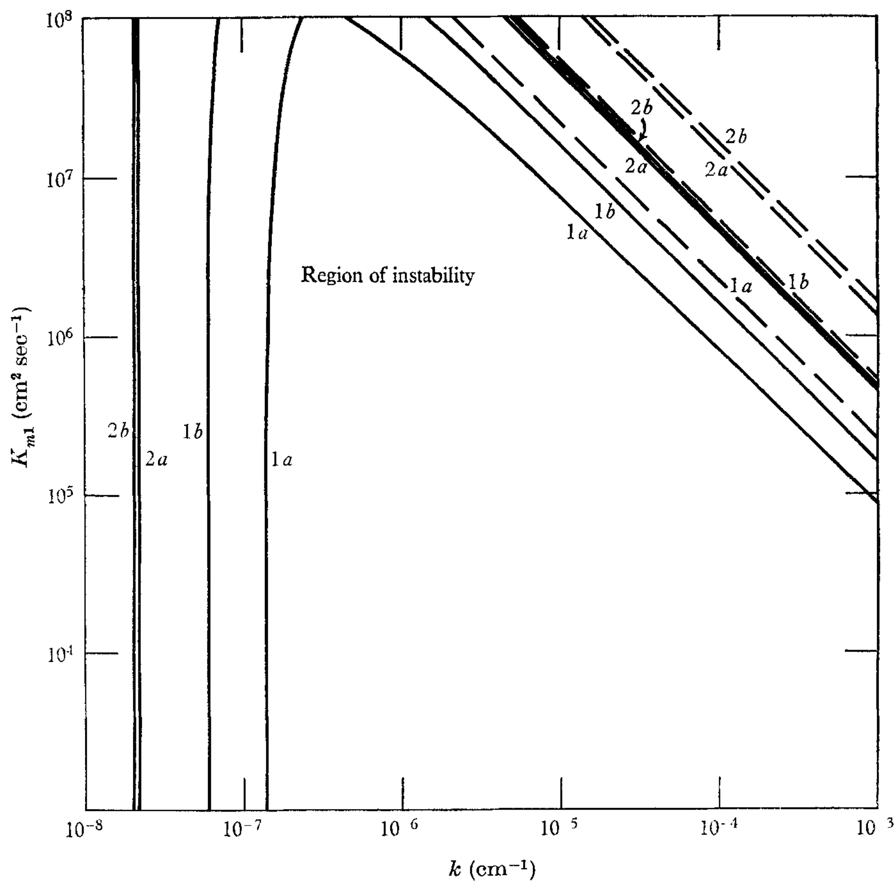

FIGURE 5. Marginal stability curves for a two-component stratification with a dynamically unstable range. Continuous lines, $H = 1000$ m; broken lines, $H = 100$ m. Labels 1 and 2 pertain to $\epsilon = -0{\cdot}92$ and $-0{\cdot}96$, a and b to $\eta_2/\eta_1 = 0{\cdot}1$ and $0{\cdot}01$.

As long as $\eta_1 > \eta_2$ (which might presumably also be the case for eddy coefficients) such a range of ϵ will exist even when the mixing is done by turbulent processes. Conditions in which internal waves will be unstable might then occur in an ocean where cold relatively fresh water overlies a warmer, more saline, water mass. Internal waves with length such as to satisfy (36) would be amplified in such surroundings. Their first influence would be to increase the intensity of the 'internally' generated turbulence, increasing the eddy coefficients and thus growing at a progressively slower rate until the instability conditions are no longer satisfied. A secondary effect of the transfer of energy from the density field to the stratification will be an increase in the intensity of the stratification. Because $\eta_1 > \eta_2$, the adverse temperature gradient will be gradually ironed out; this again produces a situation where (36) is no longer satisfied. The accelerated mixing in such a situation would favour the cooling of deeper water and might play some role in the formation of deep waters in the oceans.

For relatively long wavelengths $(a^2Rp/\eta_0 \gg \eta_3^2)$ the complex roots of (32) can be approximated by

$$\sigma \simeq -\tfrac{1}{2}[\eta_3 + (\epsilon\eta_1 + \eta_2)/(\epsilon+1)] \pm i(a^2Rp/\eta_0)^{\frac{1}{2}}. \tag{40}$$

In that case then the damping effects of heat and salt are directly additive and weighed according to their respective eddy diffusivities and contributions to the total stratification.

6. Rotating fluid

In a purely thermally stratified fluid the characteristic equation when Ω is not zero becomes

$$\sigma^3 + (\eta_1 + 2\eta_3)\,\sigma^2 + (\eta_3^2 + 2\eta_1\eta_3 + a^2Rp/\eta_0 + \Theta n^2\pi^2/\eta_0)\,\sigma + \eta_1\eta_3^2 + a^2Rp\eta_3/\eta_0 + \Theta n^2\pi^2\eta_1/\eta_0 = 0. \tag{41}$$

We immediately note that when heat and momentum diffuse at the same rate $(\eta_1 = \eta_3)$ equation (41) contains $(\sigma + \eta_3)$ as a factor, the resulting quadratic being of the form (27). The effect of rotation is in that case directly identifiable with an extra stability term, so that the frequency is increased accordingly.

When $\eta_1 \neq \eta_3$, there will exist complex roots of (41), corresponding to oscillatory solutions, when the cubic determinant of (41) is positive. This discriminant is itself written as a cubic in $A = a^2Rp/[(\eta_1 - \eta_3)^2\,\eta_0]$, with coefficient functions of $B = \Theta n^2\pi^2/[(\eta_1 - \eta_3)^2\,\eta_0]$:

$$A^3 + A^2(3B - \tfrac{1}{4}) + A(3B^2 - 5B) + (B^3 + 2B^2 + B). \tag{42}$$

This latter expression always has a negative real root, and will be negative for a range of positive values of A if there is also a pair of positive real roots. In order to prevent this, the cubic discriminant of (42) must also be positive, this being expressed by the simple inequality

$$B \geqslant \tfrac{1}{27}. \tag{43}$$

Rotation then has a general stiffening effect on the fluid, allowing, by (43), internal waves to exist even when the factor A is below the limiting value $(A = \frac{1}{4})$ for the existence of internal waves in a non-rotating fluid (equation (29), §5*a*). This result may not however be extended to apply to the case $\Gamma = 0$. In that case, only inertial waves are possible (Chandrasekhar 1961, §23); these waves are transverse, so that the only non-zero velocity component for a horizontally propagating wave is $\overline{w}$. With boundary conditions $\overline{w} = 0$ at $z' = 0, 1$, and the fact that $\overline{u}$ and $\overline{v}$ vanish identically, it follows from the continuity equation that $\overline{w}$ is also zero everywhere; (16) is then identically satisfied, with $\overline{w} \equiv 0$.

For long waves, the complex roots of (41) are approximated by

$$\sigma \simeq -\tfrac{1}{3}(2\eta_3+\eta_1)-\tfrac{1}{6}(\eta_3-\eta_1)\,(2\Theta n^2\pi^2-a^2Rp)/(\Theta n^2\pi^2+a^2Rp) \pm i[(a^2Rp+\Theta n^2\pi^2)/\eta_0]^{\frac{1}{2}}. \tag{44}$$

This approximation has been used, where applicable, to calculate the relative damping lengths x_e/L for the same values of the parameters as in figures 2 and 3 and with a rate of rotation equal to that of the Earth at the poles (figures 6 and 7). Note that the influence of rotation is strongest at long wavelengths,

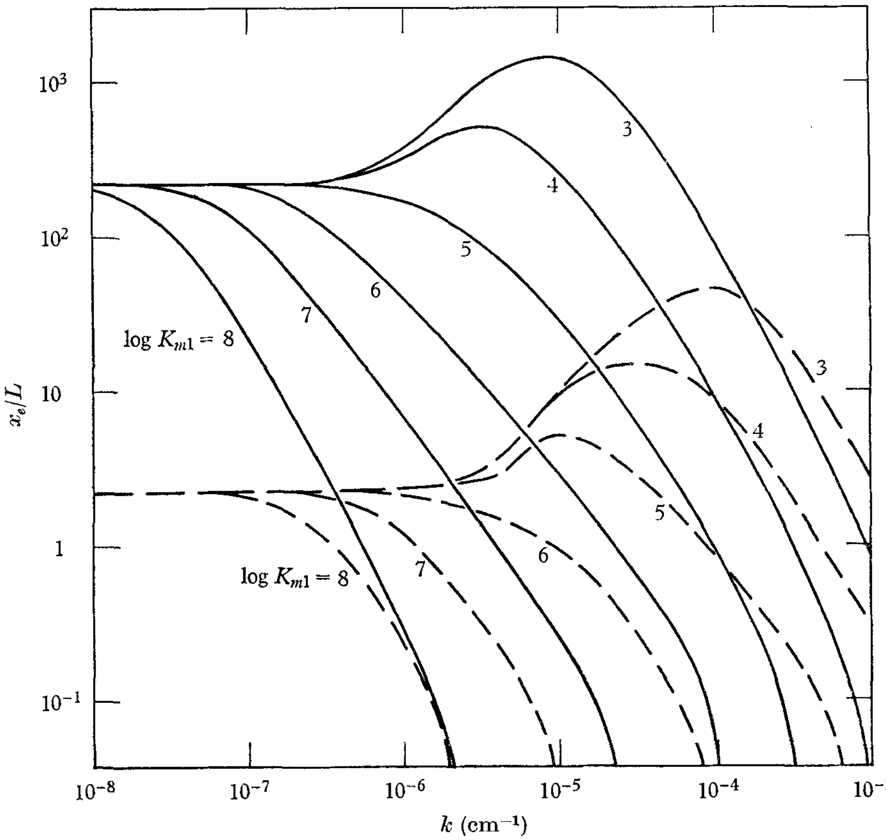

FIGURE 6. The relative damping length for $\Omega = 0{\cdot}707\times10^{-4}$ sec^{-1}, $\Gamma_2 = 0$, $n = 1$. Continuous lines, $H = 1000$ m; broken lines, $H = 100$ m.

where $\Theta n^2\pi^2/(a^2Rp) \gg 1$, and that the long wavelength cut-off which exists in the absence of rotation disappears completely (compare figure 3, $H = 10^4$ cm, and figure 7, same depth). The relative damping length is very much increased for long waves and the possible filtering action pointed out earlier correspondingly diminished. The short wave cut-off and the relative damping lengths are not much affected by rotation.

Because of the negligible influence of friction on the frequency at small wave-numbers, the long waves are still limited to periods shorter than a half pendulum-

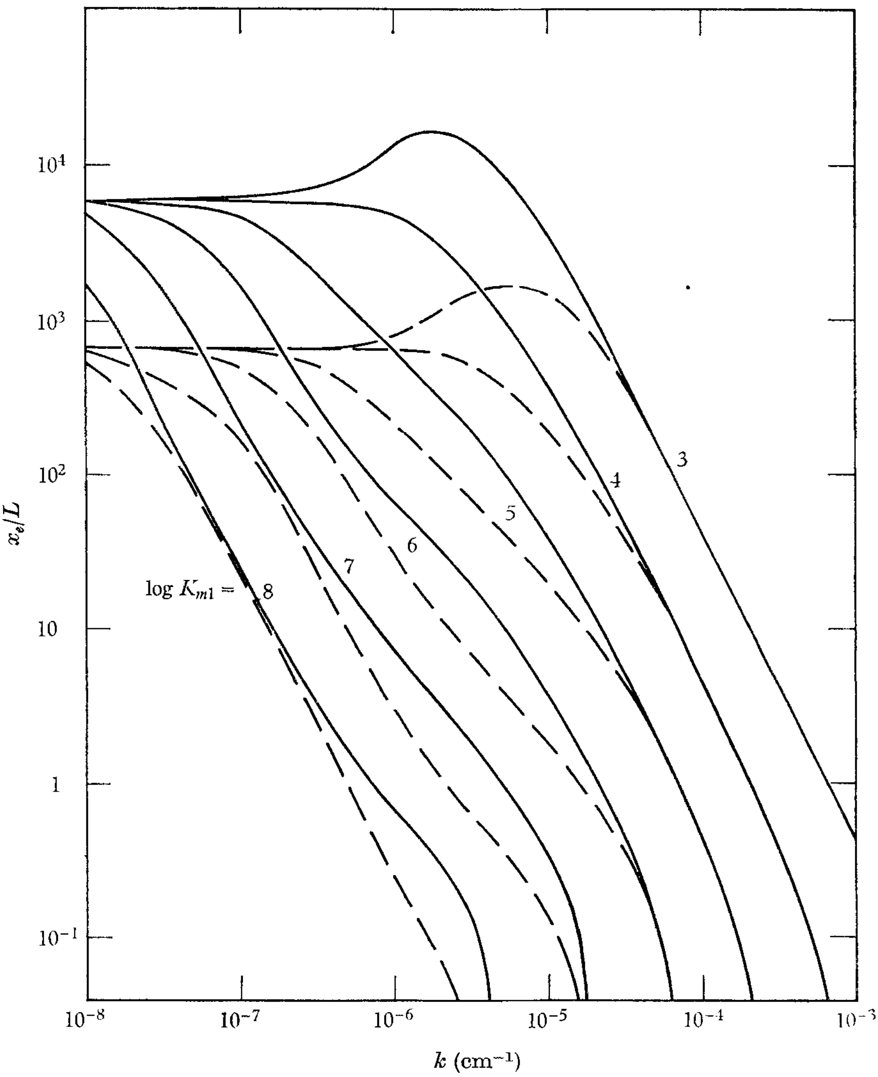

FIGURE 7. The relative damping length for $\Omega = 0{\cdot}707 \times 10^{-4}$, $\Gamma_2 = 0$, $H = 5000$ m. Continuous lines, $n = 1$; broken lines, $n = 3$.

day. Near the frequency cut-off, however, waves of very long periods are theoretically possible (figure 8). Such waves are, however, so strongly damped as to be of doubtful significance.

7. Discussion

The attenuation rates calculated above apply only to a basin with a frictionless bottom and are strongly dependent upon the magnitude of the eddy mixing coefficients. We will now impose a restriction on the magnitude of these coefficients. Since the problem was linearized, it is difficult to take into account any dependence of the strength of the turbulence upon the wave amplitude ('internal' type of turbulence); we may however use the fact that turbulent mixing processes are more important for larger scales of motion to establish some parallelism between the length of the internal waves and the magnitude of the eddy coefficients. We will not venture to postulate any functional relationship between the two but simply impose an upper bound on K_{m1}:

$$kK_{m1} \leqslant 1\,\mathrm{cm\,sec^{-1}}, \tag{45}$$

to hold in the range $10^{-8} \leqslant k \leqslant 10^{-3}\,\mathrm{cm^{-1}}$.

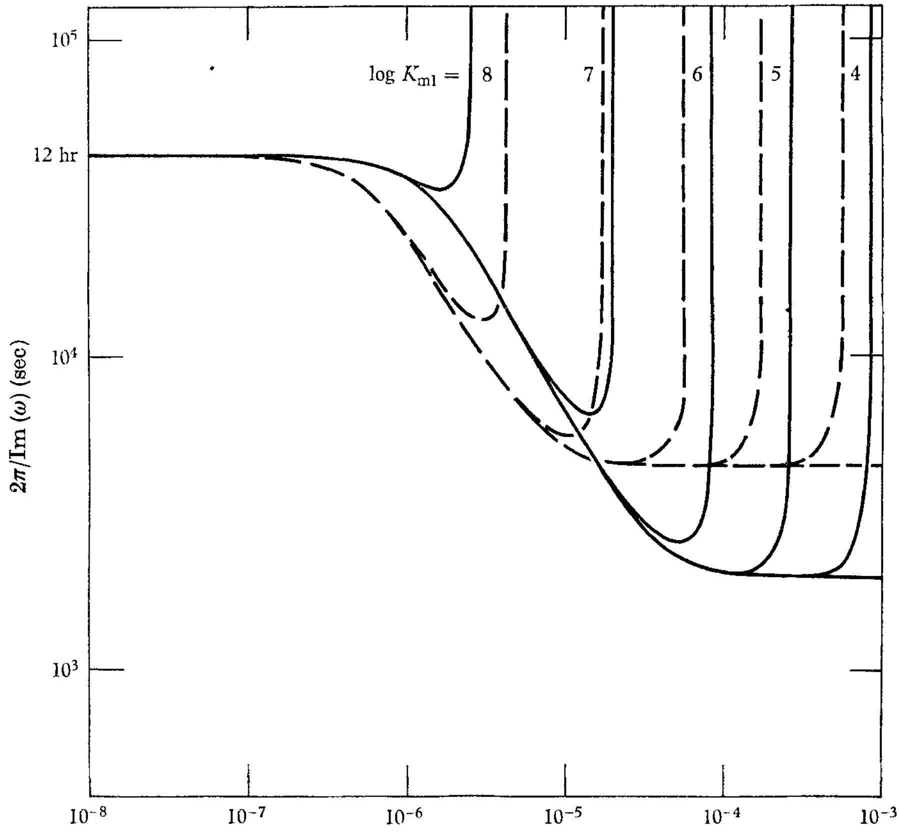

FIGURE 8. The period of the internal waves for $\Omega = 0{\cdot}707 \times 10^{-4}\,\mathrm{sec^{-1}}$, $\Gamma_2 = 0$, $n = 1$, $\Delta\rho_0/\rho_0 = 10^{-3}$ and $K_{m3} = 10^3\,\mathrm{cm^2\,sec^{-1}}$. Continuous lines, $H = 1000$ m; broken lines, $H = 5000$ m.

Similarly, on the basis of the inhibitive role of the stratification on the vertical intensity of the turbulence, an upper bound is also stipulated for K_{m3}:

$$\rho_0^{-1}\Delta\rho_0 K_{m3} \leqslant 1\,\mathrm{cm^2\,sec^{-1}}, \tag{46}$$

to hold in the range $10^{-5} \leqslant \Delta\rho_0/\rho_0 \leqslant 10^{-2}$.

For a given wave-number and strength of the stratification the damping rate, as calculated from (28), (40) or (44), according to the situation, will not exceed a maximum value corresponding to the upper bounds (45) and (46). To this maximum damping rate also corresponds a minimum relative damping length x_e/L (i.e. a minimum Q). Table 1 shows the value of this minimum Q for a

	$H = 10^4$ cm		$H = 10^5$ cm		$H = 5\times 10^5$ cm	
L	$\Omega = 0$	$\Omega = \Omega_{max}$	$\Omega = 0$	$\Omega = \Omega_{max}$	$\Omega = 0$	$\Omega = \Omega_{max}$
6280 km	—	2·2	2·6	210	47	1600
628 km	—	2·2	14·5	110	61	220
62·8 km	—	2·2	26	38	63	65
6·28 km	4·2	5·1	27	28	35	36
628 m	8·8	8·8	8·8	8·8	4	2·5
62·8 m	2·7	2·7	—	—	—	—

TABLE 1. The minimum relative damping length (x_e/L) (or minimum Q), corresponding to the maximum allowed eddy coefficients, for selected values of wavelength L, depth H, two rates of rotation ($\Omega = 0$, $\Omega_{max} = 0{\cdot}7\times 10^{-4}\,\mathrm{sec}^{-1}$), $n = 1$ and $\Delta\rho_0/\rho_0 = 10^{-3}$. Dashes are put in where $Q < 1$.

one-component stratification and some values of depth, wavelength and rotation rate. We may also note that under the restrictions (45) and (46) the frequency of the internal waves departs very little (see figure 8) from the value it would have in an ideal fluid and that the phase velocity is hardly affected by the presence of mixing.

In order to apply the results to a real ocean, it remains to make an estimate of the amount of energy lost by the waves due to bottom friction. It will be assumed that the influence of bottom friction is restricted to a thin boundary layer, of thickness δ ($|\delta| \ll H$), in which the influence of mixing terms is comparable to the time rate of change terms in (9). This particular type of boundary layer is chosen because first the problem has been linearized and the non-linear terms can therefore not enter the momentum balance, and secondly $\mathrm{Im}(\omega) > 2\Omega$, so that the time rate of change terms prevail over the Coriolis terms. Outside the boundary layer, the fluid is assumed frictionless and non-diffusive. The horizontal velocities in the boundary layer will then have the form (Lamb 1932, §345)

$$u = \overline{u}(1-\exp[(z'-1)H/\delta]), \tag{47}$$

with $\delta = (1-i)(K_{m3}/2\omega)^{\frac{1}{2}}$, and $\overline{u}$ is the wave velocity in the ideal fluid a sufficient distance above the bottom; if $|\delta|/H$ is small, $\overline{u}(z')$ is nearly equal to the maximum horizontal wave velocity, $\overline{u}(1)$. The velocities in the ideal fluid are given by

$$\left.\begin{aligned} \overline{u} &= \overline{u}_0\cos(n\pi z')\sin(kx-\omega t),\\ \overline{v} &= \overline{u}_0(2\Omega/\omega)\cos(n\pi z')\cos(kx-\omega t),\\ \overline{w} &= \overline{u}_0(n\pi/kH)\sin(n\pi z')\cos(kx-\omega t).\end{aligned}\right\} \tag{48}$$

The amount of energy dissipated by friction per unit volume is equal to the viscosity times the magnitude of the vorticity squared (Lamb 1932, §329); in this case, this is

$$\rho K_{m3}[|\partial u/\partial z|^2 + |\partial v/\partial z|^2]. \tag{49}$$

The total amount of energy dissipated by friction in the boundary layer during a cycle is then

$$\rho\overline{u}_0^2(1+4\Omega^2/\omega^2)\,\pi(\tfrac{1}{2}K_{m3}/\omega)^{\frac{1}{2}}. \tag{50}$$

On the other hand, the total average energy content of the wave is

$$\frac{\rho H}{4}\overline{u}_0^2[1+4\Omega^2/\omega^2+k^2H^2/(n^2\pi^2)]. \tag{51}$$

The Q of the waves, when damped only by the effect of bottom friction, is the ratio of the energy content to the amount of energy dissipated per cycle, that is the ratio of (51) to (50):

$$Q_{BL} = \frac{\sigma^{\frac{1}{2}}}{2\sqrt{2\pi}} \frac{(1+4\Omega^2/\omega^2+k^2H^2/(n^2\pi^2))}{(1+4\Omega^2/\omega^2)}, \tag{52}$$

which reduces, for long waves, to

$$Q_{BL} = \sigma^{\frac{1}{2}}/(2^{\frac{3}{2}}\pi). \tag{53}$$

The total amount of energy lost by the internal waves is found by adding up the contribution of the boundary layer to that of dissipation in the interior of the

	$H = 10^4$ cm		$H = 10^5$ cm		$H = 5\times10^5$ cm	
L	$\Omega = 0$	$\Omega = \Omega_{max}$	$\Omega = 0$	$\Omega = \Omega_{max}$	$\Omega = 0$	$\Omega = \Omega_{max}$
6280 km	—	0·19	0·23	$1\cdot9\times10^{-2}$	$2\cdot0\times10^{-2}$	$3\cdot9\times10^{-3}$
628 km	—	0·19	$7\cdot1\times10^{-2}$	$1\cdot8\times10^{-2}$	$6\cdot3\times10^{-3}$	$3\cdot6\times10^{-3}$
62·8 km	0·71	0·18	$2\cdot3\times10^{-2}$	$1\cdot7\times10^{-2}$	$2\cdot0\times10^{-3}$	$2\cdot0\times10^{-3}$
6·28 km	0·23	0·17	$7\cdot3\times10^{-3}$	$7\cdot3\times10^{-3}$	$8\cdot6\times10^{-4}$	$8\cdot6\times10^{-4}$
628 m	0·073	0·072	$4\cdot0\times10^{-3}$	$4\cdot0\times10^{-3}$	$7\cdot9\times10^{-4}$	$7\cdot9\times10^{-4}$
62·8 m	0·040	0·040	$4\cdot0\times10^{-3}$	$4\cdot0\times10^{-3}$	$7\cdot9\times10^{-4}$	$7\cdot9\times10^{-4}$

TABLE 2. The relative boundary-layer thickness, $|\delta|/H$, for $K_{m3} = 10^3$ cm² sec⁻¹ and the same choice of L, H and Ω as for table 1. Dashes indicate that $|\delta|/H > 1$.

	$H = 10^4$ cm		$H = 10^5$ cm		$H = 5\times10^5$ cm	
L	$\Omega = 0$	$\Omega = \Omega_{max}$	$\Omega = 0$	$\Omega = \Omega_{max}$	$\Omega = 0$	$\Omega = \Omega_{max}$
6280 km	—	1·05	1·32	4·62	4·37	22·9
628 km	—	1·05	1·36	4·65	13·8	23·9
62·8 km	—	1·05	3·98	5·11	43·8	44·4
6·28 km	1·32	1·00	13·3	13·3	352	352
628 m	1·44	1·46	235	235	$2\cdot7\times10^4$	$2\cdot7\times10^4$
62·8 m	23·7	23·7	$2\cdot2\times10^4$	$2\cdot2\times10^4$	$2\cdot7\times10^6$	$2\cdot7\times10^6$

TABLE 3. Q_{BL}, the ratio of the total average energy content of the internal waves to the amount of energy dissipated by friction in the bottom boundary layer in one cycle. $K_{m3} = 10^3$ cm² sec⁻¹; L, H and Ω as in preceding tables. Dashes indicate that $Q_{BL} > 1$.

	$H = 10^4$ cm		$H = 10^5$ cm		$H = 5\times10^5$ cm	
L	$\Omega = 0$	$\Omega = \Omega_{max}$	$\Omega = 0$	$\Omega = \Omega_{max}$	$\Omega = 0$	$\Omega = \Omega_{max}$
6280 km	—	—	—	4·50	4·00	22·6
628 km	—	—	1·24	4·46	11·2	12·7
62·8 km	—	—	3·46	4·50	25·8	26·2
6·28 km	1·01	—	8·90	9·00	31·8	32·6
628 m	1·20	1·22	8·51	8·51	4·00	2·50
62·8 m	1·26	1·26	—	—	—	—

TABLE 4. Q_T, the ratio of the total average energy content of the internal waves to the total amount of energy dissipated per cycle in both the boundary layer and the interior of the fluid. Calculated from the values of tables 1 and 3 for the values of eddy coefficients and stratification used in those tables. Dashes indicate that $Q_T > 1$.

fluid. A total Q representing the total relative dissipation rate of the waves can than be written in terms of Q_{BL} and x_e/L (which will be denoted for the purpose by Q_{int}):

$$Q_T^{-1} = Q_{BL}^{-1} + Q_{\text{int}}^{-1}. \quad (54)$$

A useful check on the applicability of the boundary-layer approximation is provided by comparing the calculated boundary-layer depth ($|\delta| = (\frac{1}{2}K_{m3}/\omega)^{\frac{1}{2}}$) with the total depth of the fluid. Table 2 gives the ratio $|\delta|/H$ for $K_{m3} = 10^3\,\text{cm}^2\,\text{sec}^{-1}$ and the same values of L, H and Ω as in table 1. It is seen that the relative boundary-layer thickness ($|\delta|/H$) is smallest for large depths and short waves, so that the boundary-layer approximation will be at its best in those conditions.

The quantity Q_{BL}, characterizing the energy dissipation rate in the boundary layer, is tabulated in table 3, for $K_{m3} = 10^3\,\text{cm}^2\,\text{sec}^{-1}$. The dependence of the dissipation rate on the wavelength is opposite to that in the interior of the fluid. The importance of bottom friction increases markedly as the depth decreases, and long waves are much more affected by it than short ones. In the latter, $n^2\pi^2 \ll a^2$ and the dissipation is mostly due to horizontal mixing.

Q_T, the ratio of the average energy content to total energy dissipation per cycle, is calculated, as for (54), from tables 1 and 3 and shown in table 4. One notices that the effect of bottom friction dominates for longer waves, that of 'interior' dissipation for shorter waves, so that the wavelength dependence of Q_T is weaker than that of either Q_{BL} or Q_{int}.

8. Conclusions

Any conclusion to be drawn from the above results on the attenuation of internal waves in the chosen model depends upon the validity of restricting the effect of bottom friction to a thin boundary layer, and upon the upper bounds chosen in (45) and (46) for the eddy coefficients. An easy check on the first of these assumptions is provided by the calculated value of $|\delta|/H$ (this is done in table 2 for one value of K_{m3}). Although the choice of upper bounds for the eddy coefficients is more subject to discussion, it still seems permissible to make the following general remarks on the behaviour of internal waves in a weak exponential stratification.

(1) Long internal waves in shallow ($H \simeq 10^4\,\text{cm}$) basins will be very rapidly damped. Even though the boundary-layer approximation may be very poor in such conditions, the damping rate in the interior of the fluid is high enough to insure rapid attenuation. Long internal seiches of the Baltic Sea, or of Hudson Bay, for example, should not last more than a few cycles after the disappearance of the exciting mechanism.

(2) Long internal tides will be slowly, but significantly, attenuated as they propagate across deep oceanic basins. For example, in an ocean 5 km deep where $\Delta\rho_0/\rho_0 = 10^{-3}$, the semi-diurnal internal tide has wavelengths of 222 and 450 km at latitudes 0° and 60° respectively, and corresponding values of Q_T of 15·7 and 17·4, and of x_e of 3500 and 7800 km. Similarly, the diurnal tide has a wavelength of 444 km at the equator, a Q_T of 12 and x_e of 5350 km, in the same conditions. Internal waves of tidal frequencies should then be observable very far from the coasts (assuming that they are generated there). On the other hand,

the attenuation rates are certainly large enough to prevent the establishment of standing wave systems. Another obstacle to large standing waves is of the great increase in damping rate suffered by an internal wave as it enters shallow waters.

(3) For waves of medium length, the damping length is in all cases much smaller than the lateral oceanic dimensions. Since such waves have been observed far away from the coasts (Ufford 1947), it must be concluded that they are generated locally, presumably through interaction with the atmosphere.

(4) Very short ($L < 100\,\mathrm{m}$) internal waves are strongly damped over the whole range of depths considered here ($100\,\mathrm{m} \leqslant H \leqslant 5000\,\mathrm{m}$). The damping is produced in this case mostly through lateral mixing in the interior of the fluid.

(5) For eddy coefficients subject to restrictions (45) and (46), the frequency of damped internal waves departs little from that of undamped waves, and the relative damping depth x_e/L depends almost exclusively on the real part of σ.

For long internal waves the present results do not differ very much from those derived by Rattray (1957) for a two-layer model. Since the longer waves are damped mostly through the effect of bottom friction, the form of the stratification does not have a strong influence on the damping rate, and the dependence on the eddy coefficients is of the same form in both models (proportional to $\sqrt{K_{m3}}$). Here, as in Rattray, the results may be interpreted to conclude that large internal tides will be concentrated in a band surrounding the continents. The width of this band will depend of course on the choice made for the magnitude of the eddy coefficients. Because of the dominance of bottom friction, and in view of the similarity of the results obtained for the two extreme cases studied so far, it seems, finally, that the damping rate of internal tides depends very little on the actual form of the stratification present in the fluid.

This work was accomplished during tenure of a Post-doctoral Fellowship from the National Research Council of Canada, held at the Institut für Meereskunde, Kiel, Germany.

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